

Thermodynamics of the $O(4)$ linear and nonlinear models within the auxiliary field method*

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The study of the $O(N)$ model at nonzero temperature is presented applying the auxiliary field method, which allows to obtain a continuous transformation between the linear and the nonlinear version of the model. In case of explicitly broken chiral symmetry the order of the chiral phase transition changes from crossover to first order as the vacuum mass of the σ particle increases. In the chiral limit one observes a first order phase transition and the Goldstone's theorem is fulfilled.

PACS numbers: PACS numbers: 11.10.Wx, 11.30.Rd, 12.39.Fe.

1. Introduction

Scalar models with orthogonal symmetry are applied in many areas of physics, like quantum dots and high-temperature superconductivity. In three spatial dimensions no analytical solution exists, therefore it is instructive to compare different many-body approximation schemes to estimate their physical relevance. In the literature the optimized perturbation theory [1], the 2PI formalism [2, 3], and the $1/N$ expansion [4, 5, 6, 7] have been used several times to examine the thermodynamical behavior of the $O(N)$ model.

In this work we study the thermodynamics of the $O(N)$ model by introducing an auxiliary field. To calculate the effective potential, the masses and the condensate at nonzero T we apply the so-called two-particle irreducible (2PI) or Cornwall-Jackiw-Tomboulis (CJT) formalism [8, 9] in the double-bubble approximation. Within the auxiliary field method the nonlinear version of the model is given by a mathematically well-defined limiting process of the linear $O(N)$ model. We find that the gap equations for the order parameter and the masses of σ and π quantitatively differ from the

* Presented at the Workshop “Excited QCD 2011” , 20-25 February 2011-Les Houches

standard treatment of the $O(N)$ model without the auxiliary field. This paper is based on the results of Ref. [10].

2. The $O(N)$ model

The generating functional at finite temperature of the linear $O(N)$ model is given by

$$Z_L[\varepsilon, h] = N \int \mathcal{D}\alpha \mathcal{D}\Phi e^{\int_0^\beta d\tau \int_V d^3x \mathcal{L}_{\sigma-\alpha}}, \quad (1)$$

$$\mathcal{L}_{\sigma-\alpha} = \frac{1}{2} \partial_\mu \Phi^t \partial^\mu \Phi - U(\Phi, \alpha) \quad U(\Phi, \alpha) = \frac{i}{2} \alpha (\Phi^2 - v_0^2) + \frac{\varepsilon}{2} \alpha^2 - h\sigma.$$

Here $\Phi^t = (\sigma, \pi_1 \dots \pi_N)$; α is an auxiliary field serving as a Lagrange multiplier. By integrating it out we obtain:

$$Z_L[\varepsilon, h] = \int \mathcal{D}\Phi e^{\int_0^\beta d\tau \int_V d^3x \left[\frac{1}{2} \partial_\mu \Phi^t \partial^\mu \Phi - \frac{1}{2N\varepsilon} (\Phi^2 - v_0^2)^2 + h\sigma \right]}. \quad (2)$$

The tree level potential exhibits now the typical “Mexican hat” shape, where $1/\varepsilon$ is the coupling constant, h the parameter for explicit symmetry breaking, and v_0 is the vacuum expectation value (v.e.v.). The advantage of the auxiliary field representation of the *linear* version of the model, Eq. (2), is that by taking the limit $\varepsilon \rightarrow 0$ one naturally recovers the *nonlinear* version of the model. Note, the limit $\varepsilon \rightarrow 0$ corresponds to an infinitely large coupling constant. In the nonlinear case the dynamics of the fields is constrained on the chiral circle, defined by the condition $\Phi^2 = v_0^2$, which is represented by a δ -function

$$Z_{NL}[h] = \lim_{\varepsilon \rightarrow 0^+} Z_L[\varepsilon, h] = \int \mathcal{D}\Phi \delta(\Phi^2 - v_0^2) e^{\int_0^\beta d\tau \int_V d^3x \left(\frac{1}{2} \partial_\mu \Phi^t \partial^\mu \Phi + h\sigma \right)}. \quad (3)$$

Here we have used the mathematically well-defined (i.e., convergent) representation of the functional δ -function

$$\delta(\Phi^2 - v_0^2) = \lim_{\varepsilon \rightarrow 0^+} N \int \mathcal{D}\alpha e^{-\int_0^{1/T} d\tau \int_V d^3x \left[\frac{i}{2} \alpha (\Phi^2 - v_0^2) + \frac{N\varepsilon}{8} \alpha^2 \right]}. \quad (4)$$

In some previous studies of the nonlinear $O(N)$ model [5, 6] the ε -dependence of the δ -function was not properly handled, since the ε -dependent term, $\varepsilon \alpha^2$, was neglected. This is, however, not correct, since this term is essential to construct the link between the linear and the nonlinear versions of the model. Besides, an integration over the auxiliary field does not give the correct potential of the linear model when this term is absent.

3. The effective potential and gap equations

The effective potential within the CJT formalism is given by

$$V = U(\phi) + \frac{1}{2} \int_k [\ln G^{-1}(k) + D^{-1}(k; \phi)G(k) - 1] + V_2(\phi, G) . \quad (5)$$

Here $U(\phi)$ is the tree-level potential, $D(k; \phi)$ the tree-level propagator in momentum space, $G(k)$ the full propagator in momentum space, and $V_2(\phi, G)$ contains all two-particle irreducible diagrams. In our case the tree-level potential is given by

$$U = -\frac{i}{2}(\alpha_0 + \alpha)(\sigma^2 + \pi_i^2 + 2\sigma\phi + \phi^2 - v_0^2) - \frac{N\varepsilon}{8}(\alpha_0 + \alpha)^2 + h(\phi + \sigma) , \quad (6)$$

where the fields σ and α have been shifted around their non-vanishing vacuum expectation values, $\sigma \rightarrow \phi + \sigma$ and $\alpha \rightarrow \alpha_0 + \alpha$. These shifts generate a bilinear mixing term, $i\alpha\sigma\phi$, rendering the mass matrix non-diagonal in the fields σ and α . Performing a further shift of the auxiliary field α , $\alpha \rightarrow \alpha - 4i\phi\sigma/N\varepsilon$, this unphysical mixing can be eliminated. The resulting Lagrangian contains no 4-point vertices

$$\begin{aligned} \mathcal{L}_{\sigma-\alpha} = & \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi_i\partial^\mu\pi_i - \frac{\sigma^2}{2} \left(i\alpha_0 + 4\frac{\phi^2}{N\varepsilon} \right) - \frac{\pi_i^2}{2} (i\alpha_0) \\ & - \frac{1}{2} \frac{N\varepsilon}{4} \alpha^2 - \frac{i}{2} \alpha(\sigma^2 + \pi_i^2) - \frac{2\phi}{N\varepsilon} \sigma(\sigma^2 + \pi_i^2) \\ & - \frac{i}{2} \alpha_0(\phi^2 - v_0^2) - \frac{N\varepsilon}{8} \alpha_0^2 + h\phi . \end{aligned} \quad (7)$$

Therefore, if we restrict ourselves to the so-called double-bubble approximation where the self-energy of the particles is independent of momentum, the contribution of V_2 to the CJT effective potential vanishes.

The gap equations are derived by minimizing the effective potential and read:

$$\begin{aligned} h = & \phi \left[M_\pi^2(\varepsilon, h) + \frac{4}{N\varepsilon} \int_k G_\sigma(k) \right] , \quad M_\sigma^2(\varepsilon, h) = M_\pi^2(\varepsilon, h) + \frac{4\phi^2}{N\varepsilon} , \\ M_\pi^2(\varepsilon, h) = & \frac{2}{N\varepsilon} \left[\phi^2 - v_0^2 + \int_k G_\sigma(k) + (N-1) \int_k G_\pi(k) \right] . \end{aligned} \quad (8)$$

4. Results

The numerical results are presented for $N = 4$ corresponding to a system of three pions and their chiral partner, the σ particle. We apply the trivial

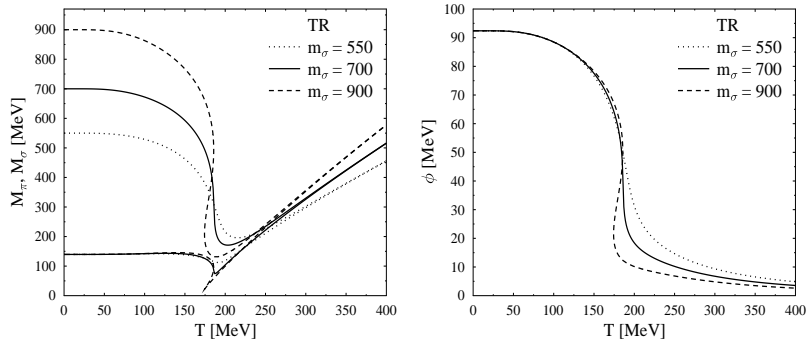


Fig. 1. The pion mass, the sigma mass and the condensate as a function of T in the $O(4)$ linear model in case of explicitly broken symmetry for different values of m_σ .

renormalisation (TR), where the divergent vacuum contributions of the tadpole diagrams is set to zero.

In the linear version of the model and for explicitly broken chiral symmetry, the order of the chiral phase transition depends sensitively on the vacuum mass of the σ particle, m_σ , see Fig. 1. Increasing m_σ the phase transition changes from crossover to first order. The identification of the chiral partner of the pion is under debate, e.g. Refs. [11].

Performing the nonlinear limit, $\varepsilon \rightarrow 0$, one observes a first order phase transition for explicitly broken chiral symmetry with the critical temperature $T_c = 178.6$ MeV, see Fig. 2. In the chiral limit, the phase transition is again of first order, see Fig. 3, with $T_c = \sqrt{12/N} f_\pi = \sqrt{3} f_\pi$, where $f_\pi = 92.4$ MeV is the pion decay constant. In the phase where the symmetry is spontaneously broken the pions are massless. Thus the Goldstone's theorem is respected. Note that from the second equation in (8) the following relation $1/\varepsilon = (m_\sigma^2 - m_\pi^2)/\phi^2$ can be obtained. Thus, the nonlinear limit is equivalent to sending m_σ to infinity.

5. Conclusions

The study of the $O(N)$ model at nonzero T was presented using the auxiliary field method to construct a mathematically well defined link between the linear and nonlinear versions of the model. To derive the thermodynamic quantities like the effective potential, the temperature dependent masses and the condensate we applied the CJT formalism in the double-bubble approximation. Although qualitatively similar to the standard double-bubble

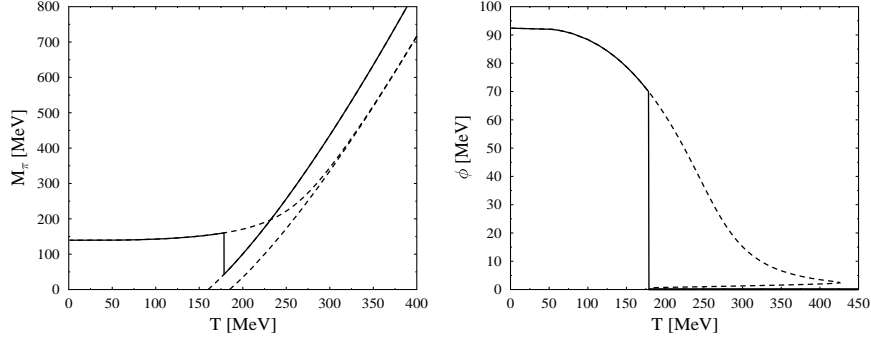


Fig. 2. The pion mass and the condensate as a function of T in the $O(4)$ nonlinear model in case of explicitly broken symmetry for $m_\sigma \rightarrow \infty$ (in practice $m_\sigma = 250$ MeV is used).

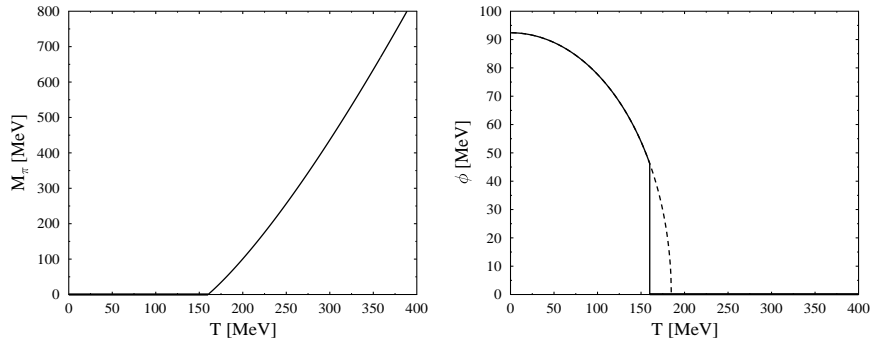


Fig. 3. The pion mass and the condensate as a function of T in the chiral limit for $m_\sigma \rightarrow \infty$ (in practice $m_\sigma = 250$ MeV is used).

approximation in the treatment without auxiliary field, the gap equations are quantitatively different and lead to different results for the order parameter and the masses of the particles as a function of T .

A natural next step is to include sunset-type diagrams in the 2PI effective action, which lead to nonzero imaginary parts for the self-energy of the quasiparticles and, in turn, to a nonzero decay width. Another project is to extend the studies to nonzero chemical potentials [7] or to include additional scalar states, since the nature of their constituency is quite unclear [12]. Besides, the application of the auxiliary field method should also be instructive for more complicated systems incorporating additional vector

and axial vector mesonic degrees of freedom [13] .

Acknowledgement

The author thank S. Strueber, F. Giacosa, D. H. Rischke, T. Brauner, M. Grahl, A. Heinz, S. Leupold, and H. Warringa for interesting discussions. The work of E. Seel was supported by the Helmholtz Research School “H-QM”.

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